Non-Homogeneous Linear Differential Equations Differential Equations X. Du

- Second-Order Linear Non-Homogeneous Differential Equation
  - Takes the form $y''+P(x)y'+Q(x)y = f(x)$
    - $f(x)$ is the input into the model, or the driving or forcing term.
  - Associated homogeneous equation: $y''+P(x)y'+Q(x)y = 0$
    - The general solution of the associated homogeneous equation is the **complementary solution**: $y_c = c_1y_1 + c_2y_2$
  - Applications:
    - Forced mass-spring system with damping: $mx''+cx'+kx = f(t)$
    - Swinging pendulum being pushed: $\theta''+\frac{g}{L}\sin\theta = f(t)$
    - RLC circuit (or LC circuit) with a battery or current source $Lq''+Rq'+\frac{q}{C} = \varepsilon(t)$ (or $Lq''+\frac{q}{C} = \varepsilon(t)$)
- An *n*th-order non-homogeneous linear differential equation
  - Takes the form $y^{(n)} + A(x)y^{(n-1)} + ... + P(x)y'+Q(x)y = f(x)$.
  - Associated homogeneous equation: $y^{(n)} + A(x)y^{(n-1)} + ... + P(x)y'+Q(x)y = 0$
    - The general solution of the associated homogeneous equation is the **complementary solution**: $y = c_1y_1 + c_2y_2 + ... + c_ny_n$
- Theorem: The solution to a linear non-homogeneous differential equation takes the form $y = y_c + y_p$, where y_c is the complementary solution and y_p is a particular solution.
  - Prove that $y = y_c + y_p$ is a solution to a linear non-homogeneous differential equation by substitution and linearity of differentiation.
  - Prove that $y = y_c + y_p$ is the general solution with the fact that y_c is the general solution of the associated homogeneous equation by linearity.
  - Relate to first-order non-homogeneous differential equations. Think of y_c as the transient and y_p as the steady-state solution – long term behavior depends on the particular solution (as long as y_c does not "blow-up").
- How to solve:
  - Find the general solution of the associated homogeneous differential equation (i.e. the complementary solution y_c) using previous methods.
  - Find any y_p (see next page).